JPRS: 4645

25 May 1961

THEORY OF OPTIMAL PROCESSES

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JPRS: 4645

CSO: 1730-S/a

THEORY OF OPTIMAL PROCESSES

-USSR-

[Following is the translation of an article by V. G. Boltyanskiy, R. V. Gamkrelidze, and L. S. Pontryagin, Corresponding Member of the Academy of Sciences USSR, in Doklady Akademii Nauk SSSR (Reports of the Academy of Sciences USSR), Vol 110, No 1, Moscow, 1956, pp 7-10.]

In recent times, in the theory of automatic control, special attention is paid to ensure very fast control, which led to the appearance of a number of works devoted to the study of the so-called optimal processes (cf (1), where one may find the bibliography of the subject). Here we give a general approach to the study of optimal processes.

1. Formulation of the problem. Let us consider a representation of a point $x = (x^1, \ldots, x^n)$ in an n-dimensional phase space, whose equations of motion are stated in the usual manner

$$\dot{x}^i = f^i(x^1, \ldots, x^n; u^1, \ldots, u^r) = f^i(x, u), \quad i = 1, \ldots, n. \qquad (1)$$

Here, $u^1, \ldots, u^r$ are control parameters. If the control mode is known, i.e., a variable vector $u(t) = (u^1(t), \ldots, u^r(t))$ is known in an r-dimensional space, then the system (1) uniquely describes the motion of the point.

We impose the natural conditions of piecewise continuity and piecewise smoothness of the vector $u(t)$, and therefore assume that the variable vector $u(t)$ is in a constant closed domain $\bar{\Omega}$ of the space of variables $u^1, \ldots, u^r$, which is called the closure of the open domain Ω with piecewise smooth $(r - 1)$-dimensional boundary. For example, the domain Ω may be an r-dimensional cube such that $|u^i| \leq 1$, $i = 1, \ldots, r$, a half space $u^1 \geq 0$, etc. The control vector $u(t)$, satisfying the stated conditions shall be called an admissible one.

Statement of the general problem. In the phase space $x^1, \ldots, x^n$, there exist two points ξ_0, ξ_1. An admissible controlling vector $u(t)$ is to be chosen in such a way that the point from position ξ_0 should arrive at position ξ_1 after a minimum of time.

The desired control vector $u(t)$ shall be called the optimal control, the corresponding trajectory $x(t) = (x^1(t), \ldots, x^n(t))$ of

system (1) is called the optimal trajectory.

2. The necessary conditions for optimality. Let us assume that there exist the optimal directions $u(t)$ and corresponding to it, the optimal trajectory $x(t)$. The trajectory $x(t)$ satisfies the boundary conditions $x(t_o) = \xi_o$, $x(t_1) = \xi_1$. Let us assume initially that the directing vector $u(t)$ for $t_o \leq t \leq t_1$ is properly contained in the open domain Ω. It follows that for arbitrary perturbations of sufficiently small modulus $\delta u(t) = (\delta u^1(t), \ldots, \delta u^r(t))$ of the vector $u(t)$, the direction $u(t) + \delta u(t)$ shall be in the domain Ω. We shall denote $x + \delta x$ the "perturbed" (i.e., corresponding to the direction $u(t) + \delta u(t)$), the trajectory of a point with a previously stated initial condition $x(t_o) + \delta x(t_o) = \xi_o$ i.e., $\delta x(t_o) = 0$. The linear approximation equations $\delta_I x = (\delta_I x^1, \ldots, \delta_I x^n)$ for the perturbations $\delta x = (\delta x^1, \ldots, \delta x^n)$ have the form

$$\delta_I \dot{x}^i = \frac{\partial f^i}{\partial x^\alpha}\delta_I x^\alpha + \frac{\partial f^i}{\partial u^\beta}\delta u^\beta; \quad \delta_I x(t_0) = 0; \quad i = 1, \ldots, n. \tag{2}$$

As the consequence of the linearity of system (2), the points $x(t_1) + \delta_I x(t_1)$ corresponding to all, for a sufficiently small modulus, perturbations $\delta_I u(t)$ fill the domain of some linear manifold P' which passes through the point $x(t_1)$. It follows easily from the optimality of the trajectory $x(t)$, that the dimensionality of the manifold P' is at most $n - 1$, and P', generally speaking, is not tangent to the trajectory $x(t)$. Let $P(t_1)$ be some $(n - 1)$-dimensional plane which contains P' and which is not tangent to the trajectory $x(t)$. The covariant coordinates of the $(n - 1)$-dimensional plane $P(t_1)$ are denoted by $a_1, \ldots, a_n$, and then $a_\alpha \delta_I x^\alpha(t_1) = 0$.

Assume that $\varphi_j(t) = (\varphi_j^1(t), \ldots, \varphi_j^n(t))$, $j = 1, \ldots, n$ is the fundamental system of solutions of the homogeneous system corresponding to system (2), and $\|\psi_j^i(t)\|$ is a matrix which is the inverse of the $\|\varphi_j^i(t)\|$ matrix. The solution of system (2) may be expressed by

$$\delta_I x^i(t) = \varphi_\alpha^i(t)\int_{t_0}^{t}\psi_\beta^\alpha \frac{\partial f^\beta}{\partial u^\gamma}\delta u^\gamma \, d\tau, \quad i = 1, \ldots, n. \tag{3}$$

Using the equality $\quad a_\alpha \delta_I x^\alpha(t_1) = 0, \quad$ we have

$$a_\alpha \delta_I x^\alpha(t_1) = a_\alpha \varphi_\beta^\alpha(t_1)\int_{t_0}^{t_1}\psi_\gamma^\beta \frac{\partial f^\gamma}{\partial u^\nu}\delta u^\nu \, d\tau = 0.$$

Let us denote $\quad a_\alpha \varphi_\beta^\alpha(t_1) = b_\beta, \ b_\beta \psi_\gamma^\beta(t) = \psi_\gamma(t).$

then $\quad a_\alpha \delta_I x^\alpha(t_1) = \int_{t_0}^{t_1}\psi_\alpha \frac{\partial f^\alpha}{\partial u^\beta}\delta u^\beta \, d\tau = 0.$

Since $\delta u(t) = (\delta u^1(t), \ldots, \delta u^r(t))$ is an arbitrary, of sufficiently small modulus, perturbation, it follows from the last equation that the system of equations is

$$\psi_\alpha(t)\frac{\partial f^\alpha}{\partial u^i} = 0, \quad t_0 \leqslant t \leqslant t_1, \quad i = 1, \ldots, r. \tag{4}$$

The vector $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ has a simple geometrical interpretation. The point $x(t) + \delta_I x(t)$ lies in the $(n - 1)$-dimensional plane $P(t)$, in which lies the point $x(t)$ and which has the

covariant coordinate system: $\psi_1(t), \ldots, \psi_n(t)$. In particular $(\psi_1(t_1), \ldots, \psi_n(t_1)) = (a_1, \ldots, a_n)$. Using the function $\psi_i(t) = b_\alpha \psi_i^\alpha(t)$, $i = 1, \ldots, n$, we obtain the system of differential equations for $\psi_i(t)$:

$$\dot{\psi}_i(t) = -\frac{\partial f^\alpha}{\partial x^i}\psi_\alpha, \quad i = 1, \ldots, n. \tag{5}$$

Combining the systems (1), (4) and (5), we have

$$\left.\begin{aligned} &\dot{x}^i = f^i(x, u), \quad i = 1, \ldots, n; \\ &\dot{\psi}_i = -\frac{\partial f^\alpha}{\partial x^i}\psi_\alpha, \quad i = 1, \ldots, n; \\ &\psi_\alpha \frac{\partial f^\alpha}{\partial u^j} = 0, \quad t_0 \leq t \leq t_1, \quad j = 1, \ldots, r. \end{aligned}\right| \tag{6}$$

The system (6) represents the totality of the necessary conditions which the optimal direction u(t) must satisfy. u(t) is properly contained in the open domain Ω and with it are associated the optimal trajectory x(t) and the vector $\psi(t)$.

Multiplying the vector $\psi(t)$ by a suitable constant (which does not change the trajectory x(t) nor the direction u(t)), we may obtain the following condition: $\psi_\alpha(t_0) f^\alpha(x(t_0)), u(t_0)) > 0$. As the plane P(t) is not the tangent plane to the trajectory x(t), i.e. $\psi_\alpha f^\alpha \neq 0$ for any t, then at any time the inequality $\psi_\alpha f^\alpha > 0$ shall be satisfied.

Now, if one should assume that the optimal direction is in the closed domain $\bar{\Omega}$ and we consider the inequality $\psi_\alpha f^\alpha \mid_{t = t_0} > 0$ then the system (4) of the necessary conditions shall become a more general condition as below

$$\psi_\alpha \frac{\partial f^\alpha}{\partial u^\beta}\delta u^\beta \leq 0, \quad t_0 \leq t \leq t_1, \tag{7}$$

for arbitrary perturbations $\delta u^\beta(t)$, on which we have "natural constraints", which follow from the condition that $u(t) + \delta u(t) \in \bar{\Omega}$.

3. <u>The sufficient conditions of optimality (locally).</u> At this point we again assume that the direction vector u(t) is properly contained in the domain Ω and satisfies the necessary conditions (6). The equations of the second approximation $\delta_{II}x$ for the perturbation δx have the form

$$\delta_{II}\dot{x}^i = \frac{\partial f^i}{\partial x^\alpha}\delta_{II}x^\alpha + \frac{\partial f^i}{\partial u^\beta}\delta u^\beta + B^i(t),$$

$$B^i(t) = \frac{1}{2}\left[\frac{\partial^2 f^i}{\partial x^\alpha \partial x^\beta}\delta_I x^\alpha \delta_I x^\beta + 2\frac{\partial^2 f^i}{\partial x^\alpha \partial u^\beta}\delta_I x^\alpha \delta u^\beta + \frac{\partial^2 f^i}{\partial u^\alpha \partial u^\beta}\delta u^\alpha \delta u^\beta\right].$$

The point whose coordinates are $x^i(t) + \delta_{II}x^i(t) = x^i(t) + \delta_I x^i(t) + \varphi^i_\alpha(t)\int_{t_0}^{t}\psi^\alpha_\beta B^\beta d\tau$

no longer lies in the plane P(t). If the moving point has passed the plane P(t) when the motion was perturbed at time t, then the scalar product is positive,

$$\psi_\alpha(t)\,\delta_{II}x^\alpha(t) = \psi_\alpha(t)\,\delta_I x^\alpha(t) + \psi_\alpha(t)\,\varphi^\alpha_\beta(t)\int_{t_0}^{t}\varphi^\beta_\gamma B^\gamma d\tau =$$

$$\psi_\alpha(t)\,\varphi^\alpha_\beta(t)\int_{t_0}^{t}\varphi^\beta_\gamma B^\gamma d\tau = \int_{t_0}^{t}\psi_\alpha B^\alpha d\tau$$

However, if the point has not yet reached the plane P(t), then

$\psi_\alpha(t)\,\delta_{II}x^\alpha(t) = \int_{t_0}^{t}\psi_\alpha B^\alpha d\tau < 0.$ The bilinear form $\psi_\alpha\frac{\partial^2 f^\alpha}{\partial u^i\,\partial u^k}\delta u^i\,\delta u^k$ (of the variables $u^1, \ldots, u^r$), at the point $(x(t_0), u(t_0), t_0)$, is negative definite. Then the scalar product is

$$\psi_\alpha(t)\,\delta_{II}x^\alpha(t) = \int_{t_0}^{t}\psi_\alpha B^\alpha d\tau < 0$$

for arbitrary, sufficiently small modules of perturbations $\delta u(t)$ and sufficiently small difference $t - t_0$. In this case the direction u(t) and the trajectory x(t) are locally optimal, i.e., the point $x(t_0)$ may be contained in such a small neighborhood V, such that if $x(t')$ and $x(t'')$, (for $t' < t''$), are two arbitrary points on the trajectory belonging to V, then for no direction, sufficiently close to u(t), one may reach the point $x(t'')$ from the point $x(t')$ during time which is less than $t'' - t'$.

If the form $\psi_\alpha\frac{\partial^2 f^\alpha}{\partial u^i\,\partial u^k}\delta u^i\,\delta u^k$ at the point $(x(t_0), u(t_0), t_0)$ is indefinite, then (for some sufficiently general additional conditions) no direction u(t) being close to the time $t = t_0$, being properly contained in the domain Ω, may be optimal, even locally. If, however, there exist optimal trajectories through the point $x(t_0)$, then the corresponding direction vectors u(t) in the neighborhood of $t = t_0$ should lie on the boundary of the closed domain Ω.

4. <u>The Maximum Principle</u>. From system (6) and the fact that the bilinear form $\psi_\alpha\frac{\partial^2 f^\alpha}{\partial u^i\,\partial u^k}\delta u^i\,\delta u^k$ is negative definite, we have that the expression $\psi_\alpha(t)\,f^\alpha(x(t), u(t))$ reaches the respective maximum for constant vectors x(t), (t) and the variable vector u(t); for sufficiently small (with respect to modulus) perturbations $\delta u(t)$ we have this inequality

$$\psi_\alpha(t)\,f^\alpha(x(t), u(t)) \geqslant \psi_\alpha(t)\,f^\alpha(x(t), u(t) + \delta u(t))$$

for all times, provided that the equations (6) are satisfied and the bilinear form is negative definite.

The above is a special case of the discussed general principle, the principle which we call the Maximum Principle (this principle has been proved by us only for some special cases up till now):

Assume that the function $H(x, \psi, u) = \psi_\alpha f^\alpha(x\ u)$ has a maximum with respect to u for arbitrary constant x and ψ, provided that the vector u varies in the closed domain Ω. This maximum we denote by $M(x, \psi)$. If the 2n-dimensional vector (x, ψ) is a solution of the hamiltonian system

$$\left.\begin{aligned} \dot{x}^i &= f^i(x, u) = \frac{\partial H}{\partial \psi_i}, \\ \dot{\psi}_i &= -\frac{\partial f^\alpha}{\partial x^i}\psi_\alpha = -\frac{\partial H}{\partial x^i}, \end{aligned}\right\} \quad i = 1, \ldots, n, \tag{8}$$

where the piecewise continuous vector $u(t)$ satisfies the condition $H(x(t), \psi(t), u(t)) = M(x(t), \psi(t)) > 0$ for all t, then $u(t)$ is defined to be the optimal direction and $x(t)$ the corresponding optimal (locally) trajectory of system (1).

We shall assume a constant initial condition $x(t_0) = \xi_0$ and as much as possible shall endeavor to specify the initial condition $\psi(t_0) = \eta_0$. Then, the system (8) together with these initial conditions and the condition $H(x(t), \psi(t), u(t)) = M(x(t), \psi(t)) > 0$ define the set of all optimal (locally) trajectories passing through the point $x(t_0) = \xi_0$, and optimal directions $u(t)$ corresponding to these trajectories.

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Received by Editors
17 April 1956

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